Zero-Knowledge Proof of Decryption for FHE Ciphertexts

Tom Kneiphof

June 28, 2018

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- Compute some function on inputs.
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- Interactive protocol amongst n parties.
- Perform computation cooperatively (By some protocol).

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- Everybody must be online.
- Asynchronous setting.
- Large group setting.

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Semi-Trusted Authority

- Authority is trusted to know the secret inputs.
- Authority is *not* trusted to perform correct computations.

- Use fully homomorphic encryption to perform computation on encrypted inputs.
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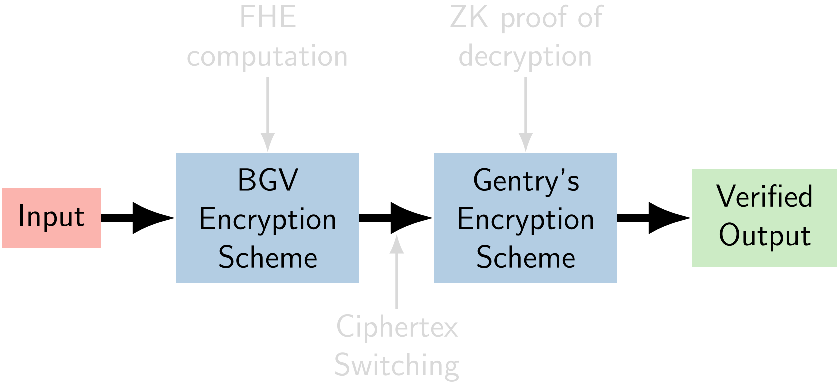
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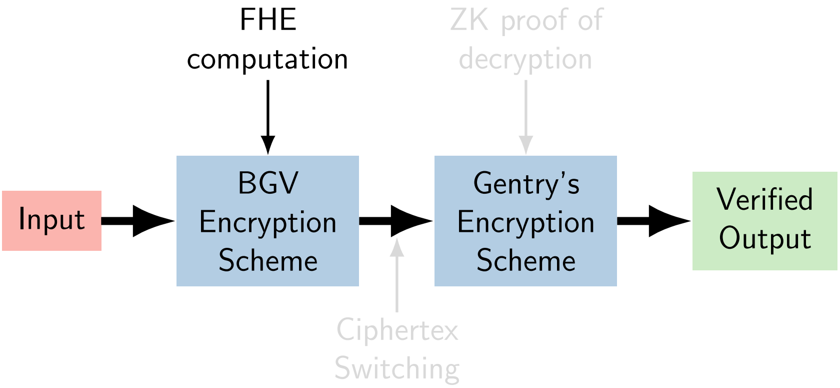
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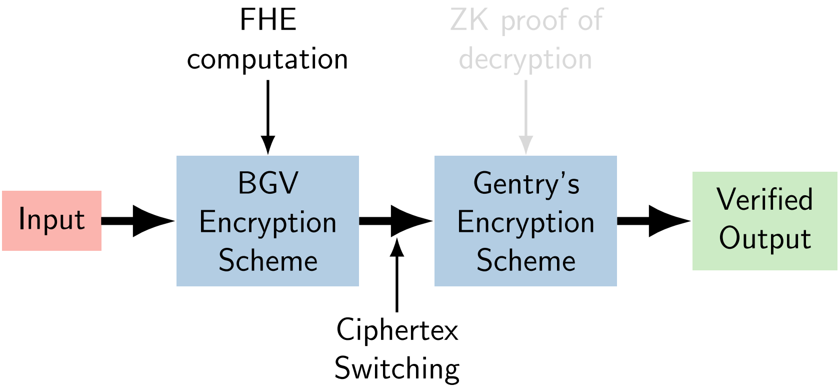
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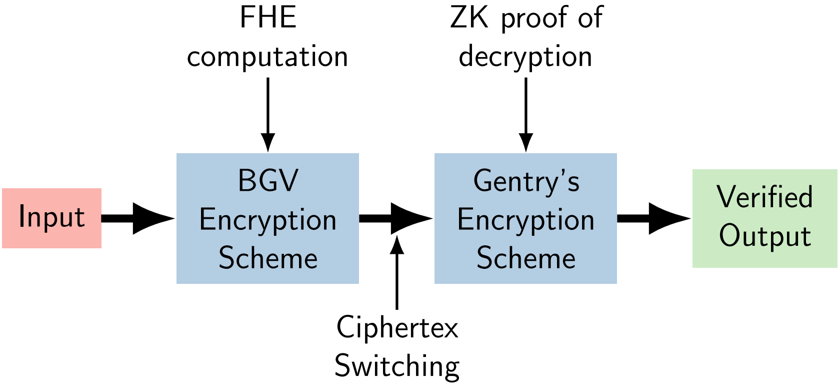
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Fully Homomorphic Encryption

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- Think of hardware circuits.
- Consist of gates (AND, OR, NAND, ...).
    - Here: Set of gates $\Gamma := \{\cdot, +\}$.
- Only consider functions that can be expressed as circuit of gates in Γ.

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- $\text{KeyGen}_{\mathcal{E}}(1^\kappa) \to (sk, pk)$.
- $\text{Encrypt}_{\mathcal{E}}(pk, \pi) \to \psi$.
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- Set of permitted circuits $\mathcal{C}_{\mathcal{E}}$.
- $\text{Evaluate}_{\mathcal{E}}(pk, C, \psi_1, \ldots \psi_t) \to \psi'$, $C \in \mathcal{C}_{\mathcal{E}}$.

Correctness:

For $C \in \mathcal{C}_{\mathcal{E}}$, plaintexts π_i and their encryption $\psi_i \leftarrow \text{Encrypt}_{\mathcal{E}}(pk, \pi_i)$, $1 \le i \le t$:

$$\psi' \leftarrow \text{Evaluate}_{\mathcal{E}}(pk, C, \psi_1, \ldots, \psi_t) \Rightarrow \text{Decrypt}_{\mathcal{E}}(sk, \psi') = C(\pi_1, \ldots, \pi_t).$$

- Ciphertext size and computation times in $\text{poly}(\kappa)$.

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- The following encryption schemes contain “noise”.
- Can decrypt $\Leftrightarrow$ Noise small.
- Homomorphic operations $\rightarrow$ Noise grows $\rightarrow$ Can't decrypt.
- “Refresh” ciphertext after homomorphic operations.

Basic Idea:

- Encrypt ciphertext under *new* key.
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Definition:

- SHE scheme $\mathcal{E}$ is circular secure, iff it is IND-CPA given encryptions of it secret key bits.

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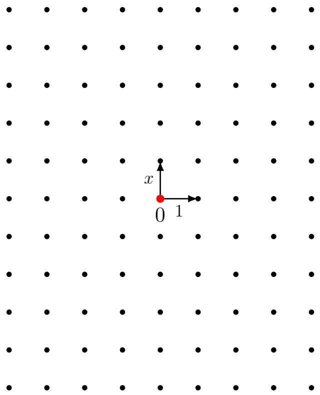

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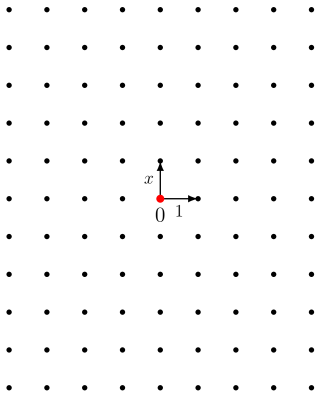

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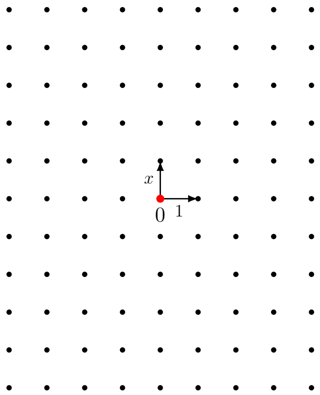

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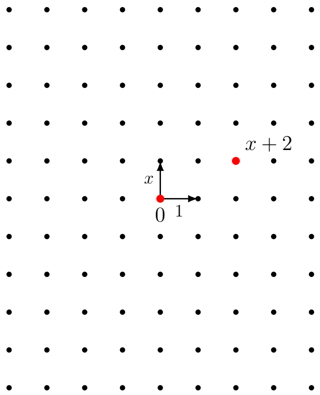

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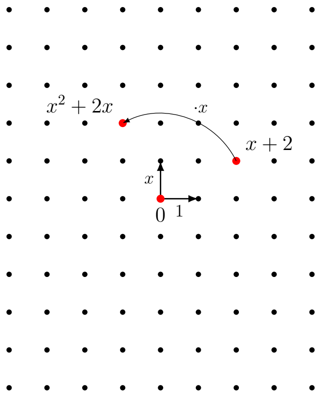

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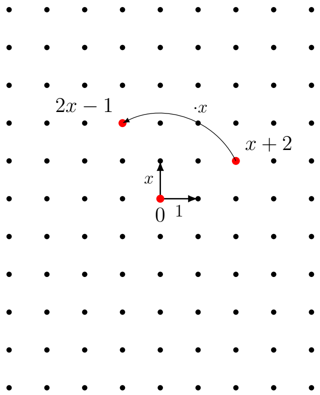

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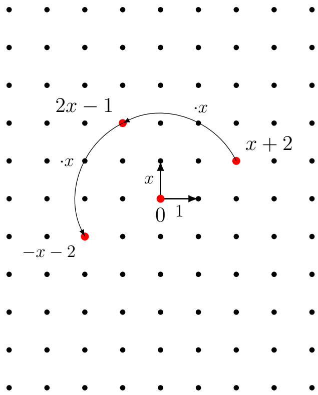

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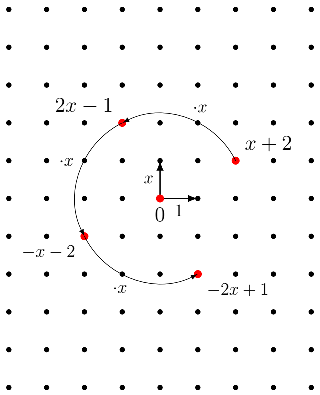

Polynomial Rings

- Common ring $R = \mathbb{Z}[x]/\Phi(x)$.
- $\Phi(x) = x^d + 1$ with $d = 2^\delta$.
- Ring of polynomials with degree at most $d-1$.
- $x^d \equiv -1 \mod \Phi(x)$.
- For $a \in \mathbb{Z}[x]/\Phi(x)$: coefficient vector $\mathbf{a} \in \mathbb{Z}^d$.
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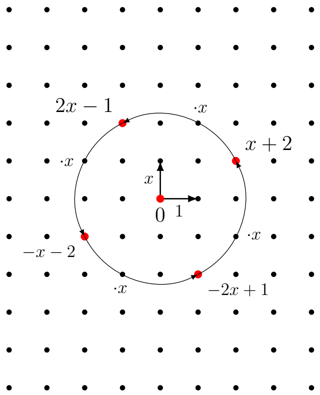

Ring Ideals

Definition:

- $I \subset R$ is called *ideal* iff for all $a, b \in I$, $r \in R$:
  - $0 \in I$.
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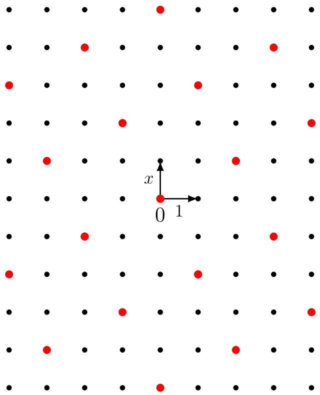

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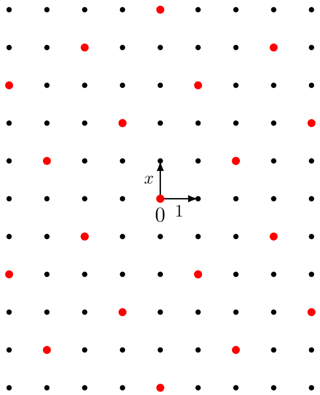

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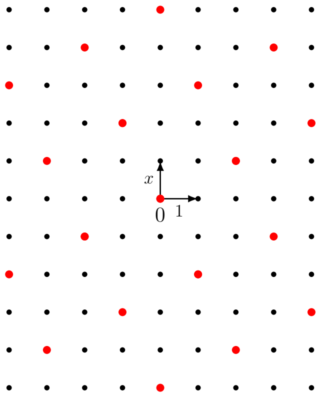

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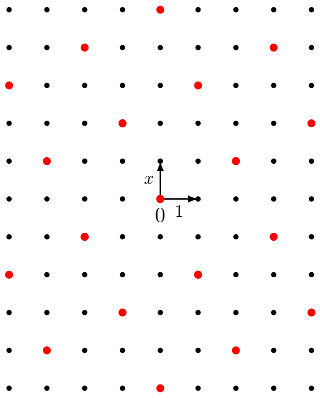
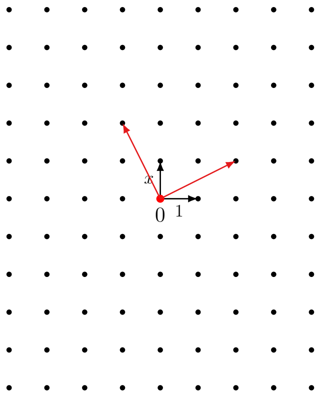


Lattices & Ideal Lattices

- Consider vector space $\mathbb{R}^d$ and some basis $\mathcal{B} \in \mathbb{Z}^{d \times d}$.
- Lattice $L = \mathcal{L}(\mathcal{B})$ is integer linear combination of columns in $\mathcal{B}$.
- Infinite number of lattice bases for $d \geq 2$.
- Ideal lattice: Ring elements corresponding to elements in L form ideal.
- Quotient ring $R/L \Leftrightarrow \mathbb{Z}^d \mod \mathcal{B}$



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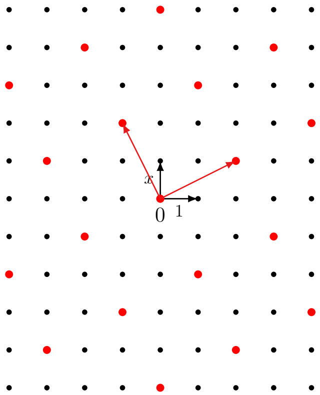

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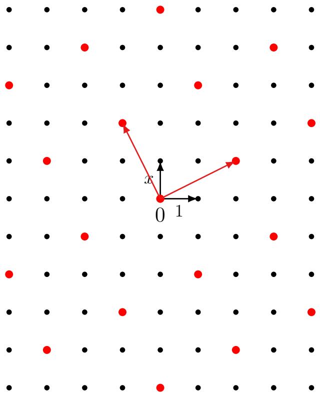

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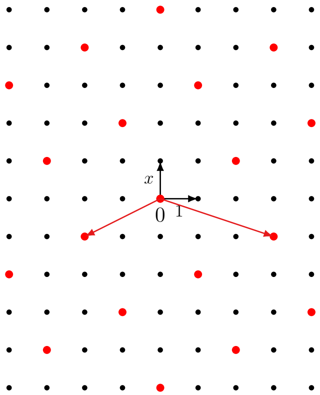

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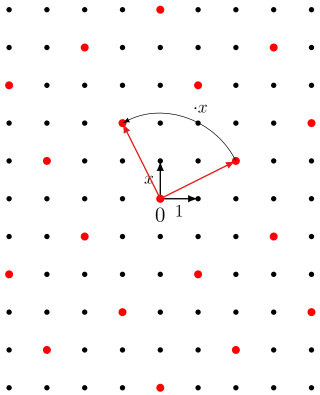

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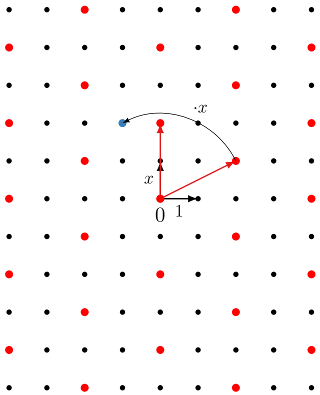

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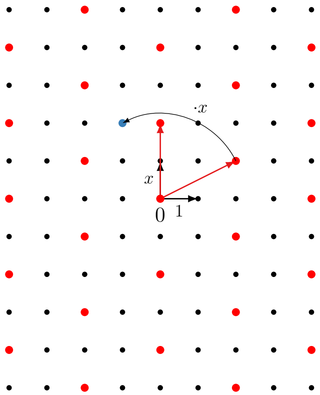

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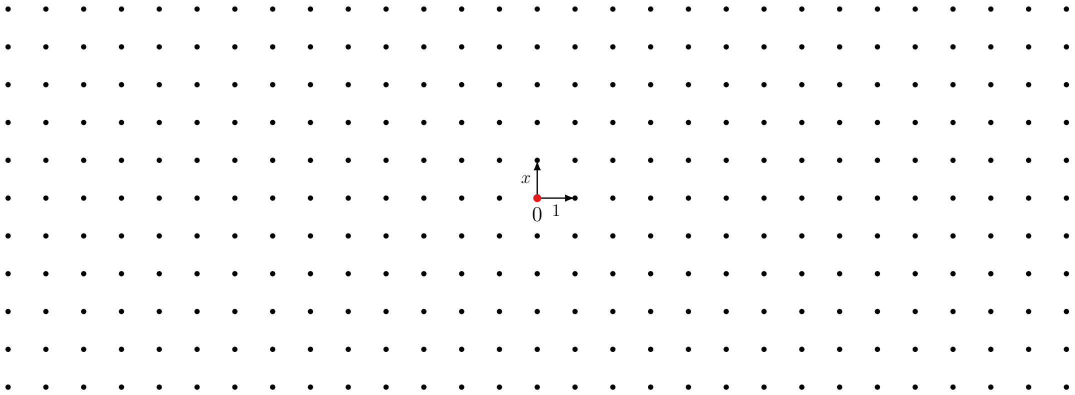

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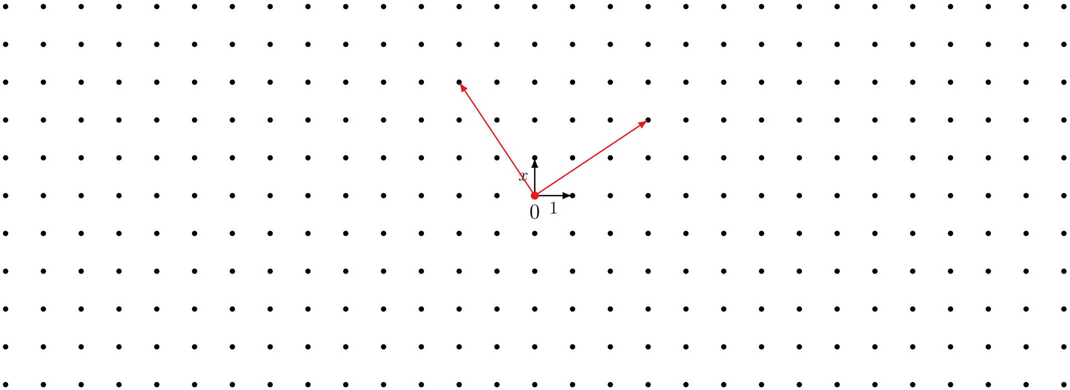

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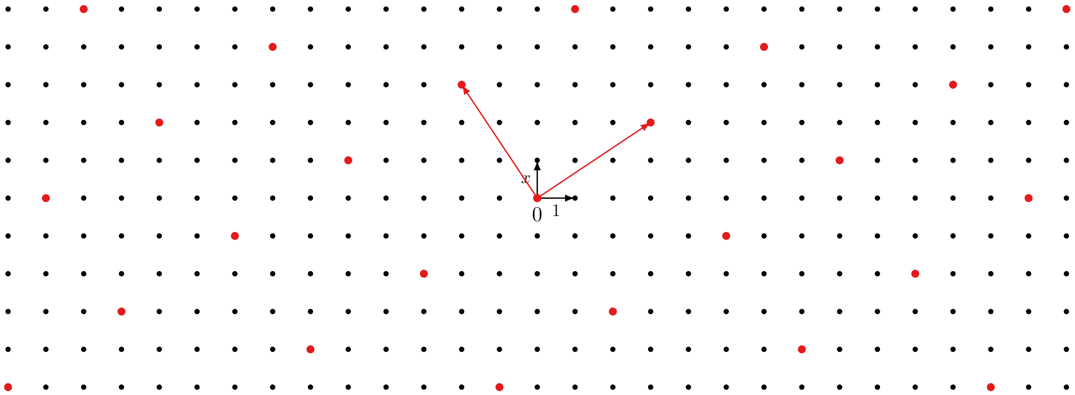

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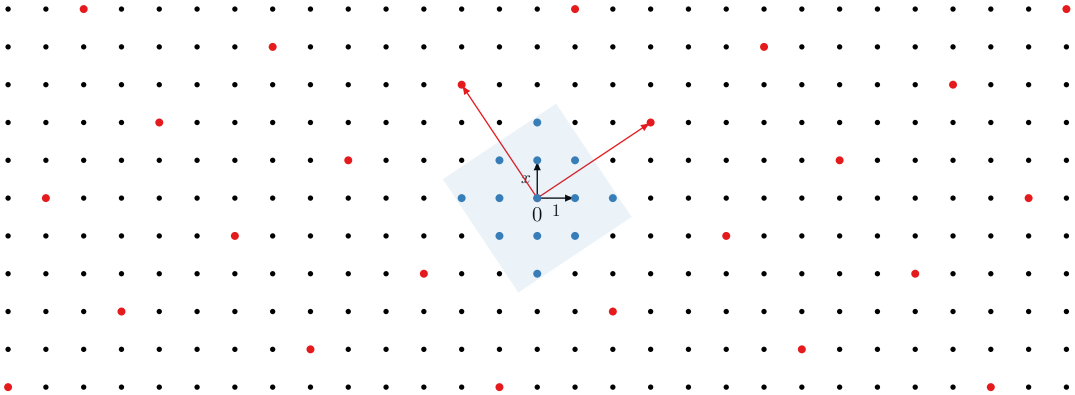

Lattice Basis - Rotation Basis

- Generating element $v \in R$.
- $\mathcal{B}_{\mathsf{Rot}}(v) = \{b_i \in R \mid b_i = v \cdot x^i\}_{i \in \{0,\ldots,d-1\}}$.

Important Feature:

- Basis vectors are (almost) orthogonal.
- $R \mod \mathcal{B}_{\mathsf{Rot}}(v)$ contains a large ball around zero.

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Definition:

- A matrix $H \in \mathbb{Z}^{d \times d}$ is in HNF, if it is a non-singular non-negative lower-triangular matrix such that each row has a unique maximum entry, which is on the diagonal.

- HNF can be computed from any basis.
- Unique HNF per lattice.

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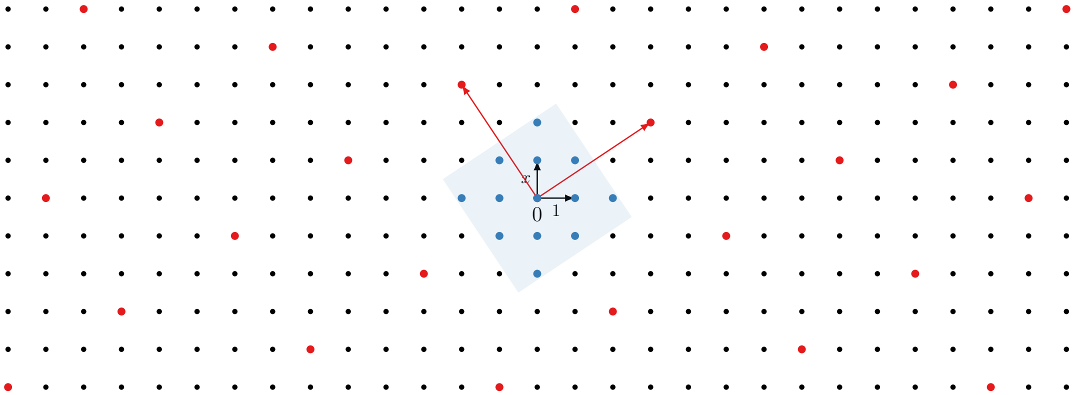
Rotation Basis vs Hermite Normal Form



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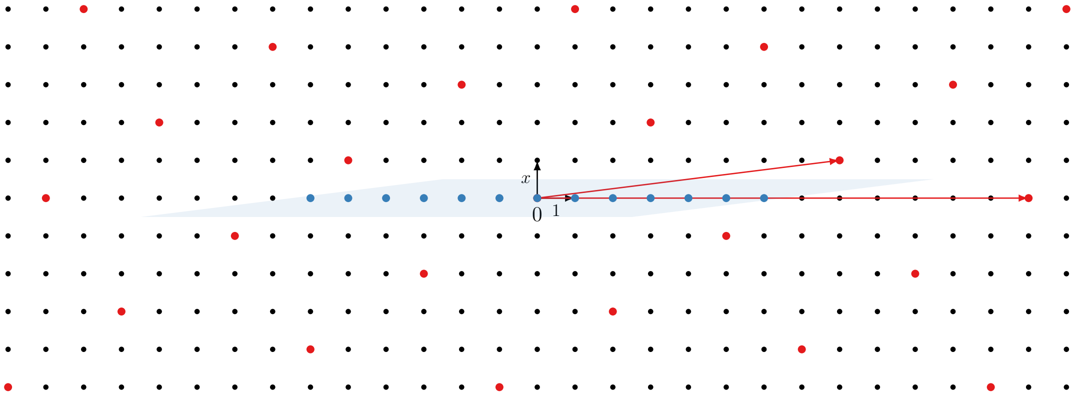

Gentry's Encryption Scheme [Gen09; GH11]

- Two ideals I and J in ring R.
- Ideal $I = 2R$ defines the plaintext space R/I.
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- A "powerful" basis is used as secret key (rotation basis).
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- Fix ring R, and basis B_I of ideal $I = 2R$.
- Generate ideal J co-prime to I and two bases (B_J^{sk}, B_J^{pk}).
- Return $pk \leftarrow (R, B_I, B_J^{pk})$ and $sk \leftarrow (R, B_I, B_J^{sk})$.

Encrypt(pk, m)

- Sample noise $rI \in I$ with $r_i \xleftarrow{\$} \{0, \pm 1\}$.
- Return $c \leftarrow m + rI \mod B_J^{pk} = m + 2r + b$ for $b \in J$.

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- Ring operations reflect operations on plaintext.
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- $c_1 \cdot c_2 = m_1 \cdot m_2 + 2(r_1 m_2 + r_1 r_2 + r_2 m_1) + b \cdot \ldots$

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- Return $sk \leftarrow \mathbf{s} = (1, s)$, $pk \leftarrow \mathbf{A} = (b, -B)$.
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BGV Encryption Scheme (simplified) [BGV14; Car+18]

KeyGen(1^κ)

- Fix ring $R = \mathbb{Z}[x]/\Phi(x)$ as before.
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BGV Encryption Scheme (simplified) [BGV14; Car+18]

Encrypt(pk, m)

- $\mathbf{m} \leftarrow (m, 0)$.
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- Return $\mathbf{c} \leftarrow \mathbf{m} + r \cdot \mathbf{A} \in R_q^2$.

Decrypt(sk, $\mathbf{c}$)

- Return $m \leftarrow \langle \mathbf{c}, \mathbf{s} \rangle \mod 2$.
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BGV Encryption Scheme - Homomorphic Operations

- Adding two ciphertexts adds their plaintext:

$$\langle \mathbf{c}_1, \mathbf{s} \rangle + \langle \mathbf{c}_2, \mathbf{s} \rangle = \langle \mathbf{c}_1 + \mathbf{c}_2, \mathbf{s} \rangle$$

- Multiplication is more difficult:

$$\langle \mathbf{c}_1, \mathbf{s} \rangle \cdot \langle \mathbf{c}_2, \mathbf{s} \rangle = \mathbf{c}_1^t (\mathbf{s} \oplus \mathbf{s}) \mathbf{c}_2 = \langle \mathbf{c}_1 \oplus \mathbf{c}_2, \mathbf{s} \oplus \mathbf{s} \rangle$$

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Ciphertext Switching

Switching Ciphertexts [Car+18]

- FHE computation: BGV scheme.
- ZK proof: Gentry's scheme.

Goal:

- Switch ciphertext via bootstrapping-like approach:
  - Encrypt BGV secret key under Gentry.
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Match Ciphertext Spaces

- Ciphertext Spaces:
  - BGV: R_q^2.
  - Gentry: $R \mod B_J^{pk}$.
- Require $qR \subset J$, i.e. $q = B_J^{pk} \cdot t$ with $t \in \mathbb{Z}^d$.
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Switch BGV Ciphertext to Gentry Ciphertext

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- BGV secret key $\mathbf{s} = (1, s) \in R_q^2$.
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- Encrypted BGV secret key $\{s\}_{\mathsf{G}}$.
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Homomorphically Decrypt BGV Ciphertext

$$\langle \{m\}_{\mathsf{BGV}}, \{\mathbf{s}\}_{\mathsf{G}} \rangle \mod B_J^{pk}$$
$$= c_0 + c_1 \cdot \{s\}_{\mathsf{G}}$$
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- $c_0 + c_1 \cdot s = m + 2e \mod q$.
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✓ Valid ciphertext if $m + 2(e + c_1 r)$ is small enough.

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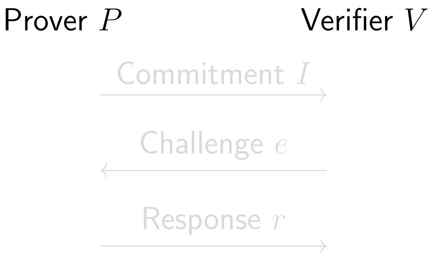
Zero-Knowledge Proof of Decryption

Sigma Protocol

- Prover P and Verifier V.
- P sends commitment I.
- V sends challenge e.
- P sends response r.
- V verifies.



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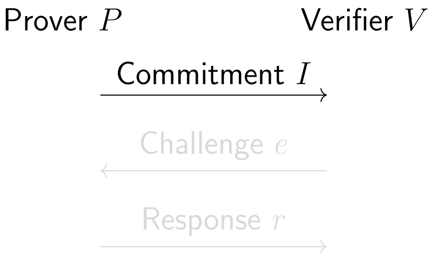

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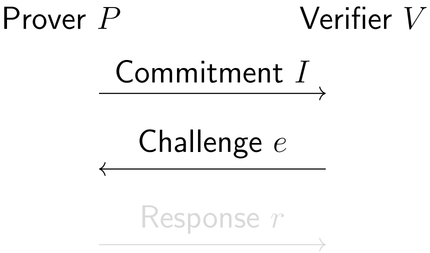

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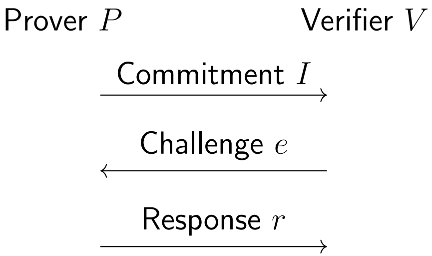

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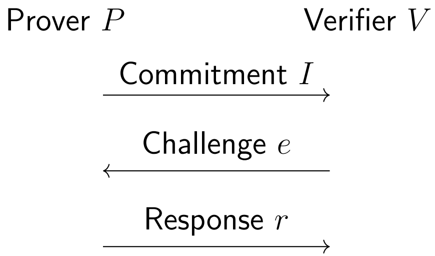

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Correctness:

- Can a true statement be proven?

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- Given two transcripts (I, e_0, r_0) and (I, e_1, r_1).
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Statement: A given ciphertext $c = m + 2r + b$ is an encryption of 0.

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If the statement is correct, then V verifies:

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$\Rightarrow$ b is a *witness* for the statement we want to prove.

Given two transcripts with same commitment: (c', e_0, d_0) and (c', e_1, d_1).

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- I.e. Simulator exists, that generates transcripts for arbitrary challenges e.

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Remarks on Fully Homomorphic Encryption Schemes

- Single bit plaintexts with current construction.
  - Parameters can be chosen to support larger plaintext spaces.
- Bootstrapping during FHE computation: Encryption uses randomness.
  - Everyone should be able to retrace computation on ciphertexts.
- Integrity during ciphertext switching?
  - Ensure that encrypted secret key during key switching is later used in ZK proof.
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- Challenge $e \in \{0, 1\}$ too simple.
  - With larger e, $e \cdot c + c'$ might be undecryptable $\rightarrow$ choose parameters wisely.
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Zero-Knowledge Proof of Decryption for FHE Ciphertexts

Thank you for your attention!

Questions?

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